NOTATIONS, PROOF PRACTICES AND THE CIRCULATION OF MATHEMATICAL OBJECTS. THE EXAMPLE OF GROUPS (1800-1860)

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How exactly do mathematical objects circulate from the work of one mathematician to another? The meaning of any mathematical text belongs to its reader: they can understand it differently from its author, they can use the ideas and concepts for a different purpose, and they can even choose different notations and symbolism to write down the mathematical objects it involves [Goldstein, 1995]. But doing so, they slightly change its original meaning. This paper is an attempt to show how far an object that would be said to remain "the same" by mathematicians working with it can be noted, symbolized and used in very different ways, and finally be associated with very different mathematical practices of proofs.

For that purpose, I will take the example of groups from the beginning of the 19th century to the 1860s. I will start from texts in which the use of groups is fundamental to the proof, but in which they are associated to a practice of proving that can be qualified of "literary": the groups do not have specific notation and are not involved into calculation. I will then analyse parts of the work of Evariste Galois (1811-1832). I will show, in particular, that symbolizing groups in different ways led him to associate them to different kind of practices of proof. I will finally examine three attempts of conceptualization that were made in the years 1850-1860 by Arthur Cayley (1821-1895), Thomas Penyngton Kirkman (1806-1895), and Richard Dedekind (1831-1916), which partly or totally relied on their readings of Galois's work, that had been published posthumously in 1846 in the *Journal de mathématiques pures et appliquées* (also known as *Liouville's journal*). In fact, these mathematicians were referring to the same corpus and were apparently talking about the same thing, namely "groups", long before it become an abstract mathematical concept, and long before a book or a treatise would give a definition of it that would be shared by a large mathematical community.

This study of the "groups in the making" will thus emphasize the diversity of approaches that mathematicians can follow when they are working with what we would be said to be *one* mathematical object. In my research about the readings and interpretations that have been made of Galois' *Mémoire sur les conditions de résolubilité des equations par radicaux*, I have studied how the way by which Galois' successors dealt with groups was linked to the specific culture they were working in [Ehrhardt, 2011a; Ehrhardt, 2012]. More precisely, each first reader of Galois endeavored to fill in the holes in Galois's proofs, but they also undertook genuine reconstructions and recastings, and endowed Galois's work with a new meaning, a mathematical « added value ». The questions that each one of those mathematicians tried to answer, the scholarly tradition into which they inscribed Galois's writings, the results to which he associates them, along with the work routines acquired in his mathematical training, the research practice and mathematical outlooks dominant in his mathematical *milieu*, and also the professional implications of his interpretation of Galois, were factors that made each of these readings different from the others. Hence, Galois's first posterity is evidence for the dependence of scholarly practice on local research traditions: mathematicians from different local traditions did not work in the same fashion and did not practice and write the same kind of mathematics.

In the present paper, I would like to emphasize that the exercise of the mathematical activity itself – the ways by which the proofs are written – is embedded into local cultures and specific historical configurations. I will then focus on *how* these mathematicians wrote the groups, on what it implied on what they were *doing* with it, and on what the group notion actually *meant* for them. This approach will allow to put to the test, in the field of mathematics, the conclusions of the anthropologist Jack Goody about lists and tables [Goody, 1977]. In particular, mathematical notations could be seen as "intellectual technologies" in the sense defined by Goody: on the one hand, they depend on specific mathematical cultures and, as so, their diffusion is socially determined; but on the other hand, mathematical notations can change the very nature of the objects, because they change the way mathematicians can use them and think about them. I hope, then, that this historical example will illustrate how different kind of mathematical notations, such as letters, lists or tables, allows mathematicians to practice different operations, to ask different questions or to solve different kind of problems, and, finally, to make mathematical proofs in very different ways. But I also hope to emphasize the fact that the different notations that can be given to a mathematical object, as well as the meanings they are supposed to express, are linked to a particular time, person or place. What we call today "the group concept" is the result a historical process of readings and transmission of papers such as the ones of Galois, Cayley and Dedekind. In this process, symbolism should be seen as a way to convey a message about the mathematical object.

1. Using groups to demonstrate in the beginning of the 19th century

One often says today that Galois had invented/discovered the group notion. Actually, as we will see in the next section, it is true that one finds the word "group" many times in his writings, may it be in the *Mémoire sur les conditions de résolubilité*, or in the letter he wrote to his friend Auguste Chevalier the day before he died, or in his rough works. But what does it exactly mean to say that "Galois has invented/discovered" the notion of group"? In fact, this attribution has a lot to do with the meaning that "groups" acquired *a posteriori* [Ehrhardt, 2011b, chap. 9]. In Galois's paper, the meaning of the word "group" was not really fixed, so that one does not find a definition of groups, in the sense we give today to a mathematical definition. As a consequence, if we want to understand what Galois meant by "group" (but also what other people of his time could understand from his works), we must examine precisely his writings, in order to find out the way he was using groups, as well as the mathematical practices that he associated to them. In other words, that is not a preliminary definition that gives a mathematical sense to groups in Galois's text; this mathematical sense comes from ways of proving, ways of writing, ways of using it.

For this reason, we need to remember that Galois was still a beginner when he wrote his research, and that he had "inherited" some mathematical practices from his training and readings. Actually, trying to "group" (that is to say to gather together in a certain way the roots of an algebraic equation in order to solve it) was already a method that mathematicians of the end of the 18th century found perfectly adequate. For instance, Lagrange wrote two works about equations: a memoir in 1770 [Lagrange, 1770] and a synthesis book in 1797, which was reedited twice at the beginning of the 19th century [Lagrange 1826]. In both cases, one can read proofs that rely on the making of specific sets of roots, organized and written in order to get a cognitive benefit about the properties of the equation, and in particular on the reasons why it will be *a priori* possible to solve it – or not. However, there is no theoretical formulation or particular conceptualization of this idea in Lagrange's published books, and the meaning of the word "group" is the one of the common language. Even if he did not quote it, one can assume that Galois knew Lagrange's synthesis book on equations [Ehrhardt, 2011b, chap. 2].

Moreover, a preface by the academician Poinsot had been added to the 1826 edition of Lagrange's book – the one available when Galois was learning mathematics. In this text, Poinsot was very precise about the mathematical use that one could make of "groups of roots". Poinso took his inspiration at the same time from Lagrange's treatise and from the seventh section of Gauss's *Disquisitiones arithmeticae* (1801):

> Les douze racines imaginaires […] se partagent en quatre groupes de trois racines, telles, dans chacun d'eux, qu'en mettant l'une à la place de l'autre, ces trois racines ne se séparent pas ; et par conséquent, si l'on échange les racines d'un groupe à l'autre, les groupes ne feront que changer de place en conservant toujours les mêmes racines. Ensuite on verra que parmi ces quatre groupes, il y en a deux qui sont tels que tout échange qui fait passer de l'un à l'autre, ramène celui-ci à la place du premier ; ainsi, les deux autres groupes sont dans le même cas ; si donc vous demandez à l'équation du $12^{\text{ème}}$ degré, le diviseur du $3^{\text{ème}}$ qui rassemblerait les trois racines d'un groupe, vous aurez les coefficients de ce diviseur par une équation du $4^{\text{ème}}$ degré ; et si vous cherchez à celle-ci le diviseur du second qui a ses racines correspondantes aux deux groupes conjugués, vous aurez ses coefficients par une équation du $2^{\text{ème}}$ degré. [Poinsot, 1808/1826, p. 370].

Here, Poinsot did much more than a mere commentary. He truly developed a new mathematical reasoning, where the Gaussian ideas of gradual solution and of grouping roots were associated to the lagrangian idea of permuting these roots. He used the same word "group" as Galois would do, and gave a lot of attention to the way this roots were placed inside these new sets.

Besides, Poinsot was far from being the only one that used the word "group" when dealing with equation solving at the beginning of the 19^{th} century. Another instance of reasoning with "groups of roots" can be find in a textbook which was of common use at Galois' time, namely Sylvestre François Lacroix's *Compléments des éléments d'algèbre*, a textbook that Galois must have also read in the context of his training at the high school Louis-le-Grand:

> En effet, parmi les 24 permutations dont ces racines sont susceptibles dans l'expression de θ, celles qui n'opéreraient que des échanges entre les valeurs de θ appartenant au même groupe ne produiraient aucun changement dans les fonctions symétriques de ces quantités. Quant aux autres permutations, elles ne feraient qu'échanger les groupes entre eux; car une valeur de θ appartenant à un groupe quelconque ne peut devenir celle d'un autre, sans que toutes les valeurs composant le premier ne deviennent celles du second, puisque les valeurs d'un même groupe se déduisent toutes de l'une quelconque d'entre elles par le changement de α en α^2, α^3 et α^4 [Lacroix, 1825, p. 49].

A third example, that Galois had certainly not read as it was unpublished at the time, is a memoir by André-Marie Ampère, written apparently in 1810:

> Mr. Poisson m'ayant communiqué deux notes extraites d'un ouvrage allemand sur la résolution des équations de tous les degrés, qui lui avaient été remises par Mr. Malus, je m'aperçus facilement que le vice de la solution indiquée dans ces notes, consistait en ce que l'auteur, après avoir dit avec raison que si l'on désigne par x, x', x'', x''', x^{IV} les racines d'une équation du cinquième degré, et a, b, c, d, e des coefficients constants, il y aura parmi les 120 valeurs dont ax + bx' +cx'' + dx''' + ex^{IV} est susceptible, 24 groupes qui répondront à autant d'équations du 5me degré dont les coefficients seront donnés par une équation du 24me degré, et qu'on obtiendra en permutant les racines sans en altérer l'ordre, et seulement en les faisant changer d'un, de deux, de trois, &c. rangs, pour avoir les cinq combinaisons d'un même groupe, prétend que si l'on réunit en une équation les coefficients de 4 de ces 24 équations du cinquième degré, en choisissant ceux qui répondent aux divers groupes qu'on forme en n'appliquant le même genre de permutation qu'à quatre des racines et laissant la cinquième à sa place, les coefficients de l'équation ainsi formés ne seront susceptibles que de six valeurs. Il faudrait pour que sans rien supposer de particulier relativement à aucune des racines, puisque l'analyse ne peut exprimer que des propriétés communes à ces cinq racines, les vingt combinaisons correspondantes à ces quatre groupes rentrassent constamment les unes dans les autres en suivant le mode de permutation convenu.
>
> Mais cela n'arrive point précisément à cause que l'on établit quelque chose de particulier à l'une des racines, en convenant d'en laisser une à sa place. Dès lors, si c'est par exemple x^{IV} qu'on laisse à sa place dans un groupe, on aura un assemblage de quatre groupes, mais si c'est ensuite x''' qu'on ne déplace pas dans le premier groupe, on en aura trois autres que le calcul lui associera nécessairement en même temps, ce qui donnera un autre assemblage de quatre groupes qui ne pourront être séparés [de?] celui qu'on avait d'abord obtenu, puisque ces deux assemblages auront un groupe commun, l'équation correspondante ne pourra donc manquer de s'élever plus haut que le 6ème degré[1].

This text shows even more precisely than the two others that the idea of "grouping" roots, as well as the idea of looking to the different ways by which these roots could be placed within each group when one permutes them, was quite known and spread at the beginning of the 19th century – at least among the people learned in and interested by mathematics. In other words, "groups" were already associated with a mathematical practice of proving results about equations. This practices consisted in making and organizing sets of roots. However, this mathematical practice took the form of an explanation, given in a literary style, instead of the one of a calculation. While using "groups of roots" in mathematical reasoning about the algebraic solution of equations, these mathematicians didn't try to symbolize the group, or to imagine a specific mathematical notation for it. Instead, they associated groups with a mental

[1] A.-M. Ampère, « Essai d'une solution complète des équations du 5e degré », Archives de l'Académie des sciences, Paris, chemise 25, carton 2.

image: they “looked like” a kind of racks where roots were placed and moved all along the proof.

These examples also show us that, even if Galois actually did do something new from these ideas about groups of roots, he also took as a point of departure things that were very well known at the time, in particular because they were part of the training of future scientists and engineers. As far as the idea of group is concerned, Galois’s mathematics were not “out of his time”, contrary to what is very often said about them. Galois made new and interesting things from mathematical ideas and practices that already existed and that he had learned (which was already an achievement!). But what did Galois do exactly with groups? Where do we find groups in his writings?

2. Writing practices associated to groups in Galois’s writings.

One of the specificity of Galois’s work is that one can read the word “groups” several times, but never with a precise definition of it. One thing that makes Galois’s writings difficult to read is that there are several “strata”, that is to say that he often re-wrote his papers; sometimes the first versions have not been preserved and are lacking; sometimes several versions still exist, but not all of them are dated. However, one can distinguish between two different kings of writing of the group notion, each of them being associated to specific uses and mathematical practices. On the one hand, Galois wrote groups with a tabular notation; on the other hand, he wrote groups with a single letter. From what I managed to reconstruct from the chronology of Galois’s writings, with the help of Bourgne and Azra’s edition [Galois, 1997], I would say that the first notation is rather linked to stages of research and clarification, while the second notation is rather linked to attempts of finding ways to formulate the results. However, both have been used by Galois from the beginning of his research to its end, that is to say from 1828 to 1832.

2. a. Group as a table

In his *Mémoire sur les conditions de résolubilité des équations par radicaux* submitted to the French Academy of Science, Galois defined the “group of an equation” in the following theorem:

> Soit une équation donnée, dont a, b, c, … sont les *m* racines. Il y aura toujours un groupe de permutations de a,b,c… qui jouira de la propriété suivante :

> 1° Que toute fonction des racines invariable par les permutations de ce groupe soit rationnellement connue ;
>
> 2° Réciproquement, que toute fonction des racines, déterminable rationnellement, soit invariable par les substitutions. [Galois, 1997, p. 51].

It is important to notice here that, if one only read the theorem alone, the word "group" doesn't necessarily have a specific mathematical sense. It could just as well be understood as a simple set of roots. Actually, the mathematical meaning comes from the proof of the theorem and, more precisely, from the way this group is written in it:

> Soit V une fonction rationnelle des racines telle que toute les racines soient fonctions rationnelles de V. Considérons l'équation irréductible dont V est une racine. Soient V, V', V'', …, $V^{(n-1)}$ les racines de cette équation.
>
> Soient φV, $\varphi_1 V$, $\varphi_2 V$, …, $\varphi_{m-1} V$ les racines de la proposée.
>
> Ecrivons les n permutations suivantes des racines.
>
> $$\begin{array}{c|cccccc} (V) & \varphi V, & \varphi_1 V, & \varphi_2 V, & \ldots., & \varphi_{m-1} V, \\ (V') & \varphi V', & \varphi_1 V', & \varphi_2 V', & \ldots., & \varphi_{m-1} V', \\ (V'') & \varphi V'', & \varphi_1 V'', & \varphi_2 V'', & \ldots., & \varphi_{m-1} V'', \\ \ldots. & \ldots\ldots & \ldots.. & \ldots.. & \ldots. & \ldots\ldots. \\ (V^{(n-1)}) & \varphi V^{(n-1)}, & \varphi_1 V^{(n-1)}, & \varphi_2 V^{(n-1)}, & \ldots., & \varphi_{m-1} V^{(n-1)} \end{array}$$
>
> Je dis que ce groupe jouit de la propriété énoncée. [Galois, 1997, p. 52]

Hence, this is this tabular disposition that actually *defines* the group of an equation. Moreover, it has three different functions. First, even if Galois never used it in the following proofs of this memoir, other documents show that he actually relied on the visual aid provided by the tables when he was thinking about groups. For instance, we can read on a rough work that must have been written in 1831:

> Groupe réductible est un groupe dans les permutations duquel n lettres ne sortent pas de n places fixes. Tel le groupe
>
> a b c d e   a b d e c   a b e c d
>
> b a c d e   b a d e c   b a e c d
>
> Un groupe irréductible, etc.
>
> Un groupe irréductible est tel qu'une lettre donnée occupe une place donnée […]
>
> Groupe irréductible non primitif est celui où l'on a *n* places et *n* lettres telles que une de ces lettres ne puisse occuper une de ces places, sans que les *n* lettre n'occupent les *n* places. [fol. 84, Galois, 1997, p. 79]

So, the spatial disposition of the elements, that is to say their "places", plays a role in the study of the properties of groups. In that case, Galois actually used the idea of "organizing roots into sets" that mathematicians like Lagrange, Lacroix and Ampère already knew, but he made a mathematical translation instead of a literary explanation. Even if this notation tended to disappear in final versions of Galois' texts, it remained a tool for reasoning in Galois's mathematical research process. It was, for instance, what he was still doing while working on elliptic functions during the year 1832, as we can see on his manuscripts things like this one [fol. 159b, Galois, 1997, p. 311][2].

$$\begin{array}{c|c|c} 4\ 3 & 2\ 1 & \infty\ 0 \\ 1\ 2 & 0\ \infty & 4\ 3 \end{array}$$

Second, the tabular notation could also be a tool to write a mathematical proof about groups. Let's take for instance an extract from the manuscript entitled "Des equations primitives qui sont solubles par radicaux":

> Cela posé, soient
>
> $$a_0\ a_1\ a_2\ \ldots.\ a_{P-1}$$
> $$b_0\ b_1\ b_2\ \ldots.\ b_{P-1}$$
> $$c_0\ c_1\ c_2\ \ldots.\ c_{P-1}$$
> $$\ldots\ldots\ldots\ldots\ldots\ldots$$
>
> Les N lettres: supposons que chaque ligne horizontale représente un système de lettres conjointes.
>
> Soient
>
> $$a_0\ a_{0.1}\ a_{0.2}\ \ldots.\ a_{0.P-1}$$
>
> P lettres conjointes toutes situées dans la première colonne verticale (il est clair que nous pouvons faire qu'il en soit ainsi, en intervertissant l'ordre des lignes horizontales).
>
> Soient de même
>
> $$a_{1.0}\ a_{1.1}\ a_{1.2}\ \ldots.\ a_{1.P-1}$$
>
> P lettres conjointes toutes situées dans la seconde colonne verticale, de sorte que
>
> $$a_{1.0}\ a_{1.1}\ a_{1.2}\ \ldots.\ a_{1.P-1}$$
>
> appartiennent respectivement aux mêmes lignes horizontales que

[2] See also fol. 177a, in [Galois, 1997, p. 331].

$$a_0\ a_{0.1}\ a_{0.2}\ \ldots.\ a_{0.P-1.}$$

Soient de même les systèmes de lettres conjointes

$$a_{2.0}\ a_{2.1}\ a_{2.2}\ \ldots.\ a_{2.P-1}$$

$$a_{3.0}\ a_{3.1}\ a_{3.2}\ \ldots.\ a_{3.P-1}$$

....

Nous obtiendrons ainsi en tout P^2 lettres. Si le nombre total de lettres n'est pas épuisé, on prendra un troisième indice, en sorte que

$$a_{m.n.0}\ \ a_{m.n.1}\ \ a_{m.n.2}\ \ldots.\ a_{m.n.P-1}$$

soient en général un système de lettres conjointes. Et l'on parviendra ainsi à cette conclusion que $N=P^{\mu}$, μ étant un certain nombre égal à celui des indices différents dont nous avons besoin. [Fol. 37b and fol. 38a, Galois, 1997, p. 131-133].

As we can see, the whole proof was based on the possibility given by the tabular representation to see analogies between the lines and the columns, and to associate a specific place to each element, what was symbolized by the indexes coordinates 0.1, 0.2, etc. The proof also strongly relied on the formal symmetry of the table, once again between lines and columns, which made it equivalent to "enter" into the table either horizontally or vertically. Moreover, the image of the table, written once at the beginning, seemed to have such a power of suggestion that the mental representation of the group could finally take the place of the written one. In fact, as we can see, Galois indicated the manipulation one had to make on the elements of the table, but he didn't need to write them effectively because it was not difficult to imagine the steps in one's head. Therefore, we could say that another function of the table was to play the role of a mental support that for the representation of the object. In that case, the tabular notation clearly allows mathematical practices that would be very difficult to express in French language.

There is still a third use of tabular representation in Galois's work that I would like to focus on, which illustrate the relation between the form and the sense of the groups in a slightly different way. The memoir on equations had been sent to the Academy of Science, which means that it was a paper that Galois had written in order that it would be be read by other mathematicians. Then, the aim was not only to do mathematical research and find convenient ways to write it down, but also to make sure that the results would be understood by the readers. Giving an example in a mathematical language and notation that people are used to, could be a good way to this end. Thus, just after the very theoretical proof of his fundamental theorem, Galois used the table notation to illustrate how his chain of reasoning worked, on the well-known case of a 4th degree equation:

> Il est aisé d'observer cette marche dans la résolution connue des équations générales du 4ème degré. En effet, ces équations se résolvent au moyen d'une équation du 3ème degré, qui exige elle-même l'extraction d'une racine carrée. Dans la suite naturelle des idées, c'est donc par une racine carrée qu'il faut commencer. Or en adjoignant à l'équation du quatrième degré cette racine carrée, le groupe de l'équation qui contenait en tout 24 substitutions, se décompose en deux qui n'en contiennent que douze. En désignant par a b c d les racines, voici l'un de ces groupes :
>
> | | | |
> |---|---|---|
> | a b c d | a c d b | a d b c |
> | b a d c | c a b d | d a c b |
> | c d a b | d b a c | b c a d |
> | d c b a | b d c a | c b d a |
>
> Maintenant ce groupe se partage lui-même en trois groupes, comme il est indiqué aux théorèmes II et III. Ainsi par l'extraction d'un seul radical du troisième degré il reste simplement le groupe :
>
> a b c d
b a d c
c d a b
d c b a
>
> Ce groupe se partage lui-même en deux groupes : a b c d c d a b
b a d c d c b a
>
> Ainsi, après une simple extraction de racines carrée, il restera : a b c d
b a d c
>
> Ce qui se résoudra enfin par une simple extraction de racine carrée. [Galois, 1997, p. 99]

Here, the table representation was used to make it easier for the reader to understand, for two reasons. On the one hand, the tables were written in such a way that it seemed very "natural" that the groups would split into smaller ones. For instance, the first line of the 12-order group was not written in alphabetical order, but in order to make it clear that the three big columns were equivalent; then, the way the group of order 4 was written "anticipated" its split into the two groups of order 2 and, finally, these two final groups were once more written in a way that made clear that they were equivalent. On the other way, using such a "combinatory method" must have been a usual practice not only for Galois, but also to his readers. As a matter of fact, the way the groups are written in this example was very close to the notations used by Lagrange in his research, which was, as we have seen, very well known of every geometers at that time[3]. So, even if it was not the same thing, the ways Galois wrote his results actually looked like something the readers were used to see. This way of writing may have made this paper look more familiar for the first French readers. From this example we could say, then, that using a specific notation is also a manner to remain within a specific mathematical culture, historically and socially determined: one employs ways of writing that one has already seen and, doing that, one makes it possible that readers sharing the same culture should understand it.

2. b. Group as a single letter

The other way used by Galois to write the group of an equation is a single letter: "le groupe G". Galois nearly didn't use it in the memoir sent to the Academy, where he much more often preferred to write "le groupe" or "le groupe de l'équation" in words. This notation didn't either appear in the papers published by Galois in the *Annales de Gergonne* and the *Bulletin de Férussac*. Moreover, as we have seen, it wasn't the way to write the groups that Galois used when he was actually doing his research, preferring in that case the table notation. But we still find it on several manuscripts, including the preliminary version of one of the theorem of his memoir (summer 1830), some rough works written at the end of 1831, and the manuscripts on primitive equations (1830). There is also an extract of the letter to Galois's friend Auguste Chevalier, written just before Galois's death in 1832, where Galois uses it. Let's look to some of these occurrences of the letter notation:

> Soit un groupe G de mt.n permutations, qui se décompose en n groupes semblables à H. Supposons que le groupe H se décompose en t groupes de m permutations semblables à k […] » [Fol. 95a, Galois, 1997, p. 149].
>
> « On appelle groupe un système de permutations tel que etc. Nous représenterons cet ensemble par G. GS est le groupe engendré lorsqu'on opère sur tout le groupe G la substitution S. Il sera dit semblable, etc. » [Fol. 84a, Galois, 1997, p. 79].
>
> « Le groupe G dont l'équation est soluble par radicaux doit se partager en un nombre premier de groupes H semblables et identiques. Ce groupe H est un nombre premier de groupes K semblables et identiques, et ainsi de suite jusqu'à un certain groupe M qui ne contiendra plus qu'un nombre premier de substitutions » [Fol. 55a, Galois, 1997, p. 97]

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[3] See for instance [Lagrange, 1770, p. 321], where Lagrange used a tabular notation to organise the roots, just like Galois did in fol. 37. See also [Lagrange, 1770, p. 394], where Lagrange wrote all the ways to permute the letters, and used a systematic process for that: he kept the same last letter and permutes the others.

> « Quand un groupe G en contient un autre H, le groupe G peut se partager en groupes, que l'on obtient chacun en opérant sur les permutations de H une même substitution ; en sorte que : G=H+HS+HS'+…. [Fol. 8a, Galois, 1997, p. 173-174].

The single-letter notation led to another way to work on groups. One obvious thing is that each single elements of the group didn't matter anymore. When using the table notation, "the unity" with which Galois was reasoning was an element, or a set of elements going together in lines or columns. Here, on the contrary, the group could be seen as one mathematical object, and not as a kind of cluster. Another thing that may be worth to notice is that, with this notation, there was no need to precise what were exactly the elements contained in the group and to explain how they "moved" from one place to another within the group. Moreover, even if these elements were always substitutions in Galois's research, the second citation is very explicit on the fact that a group could eventually be *not* attached an equation.

These two properties of the single-letter notation had direct consequences on the mathematical practices used by Galois. On the one hand, these examples show that what Galois considered that the right thing to do with groups was to split them into smaller ones. But this kind of operation would be confusing without giving a name to each of the groups, especially when there was a three-steps decomposition. On the same way, it would not have been very convenient to write a table for every group Galois was talking about. On the contrary, with the single-letter notation, the successive steps were symbolized by the alphabetic order ("G, H, K,…, M") which played, implicitly, the same role as an inclusion symbol. Moreover, the single-letter notation allows to write "H is contained in G" without writing explicitly what are the elements of G which are also in H. Hence, the single-letter notation gives the possibility to look to how a group can split into smaller ones, from a general point of view, and independently of what happens *exactly* to each element. Then, with this notation, groups become much easier to handle

On the other hand, practices of "manipulation" of groups become easier. More precisely, Galois could rely on an operating formalism analogous to the one of usual algebraic operations. In the letter to Chevalier, he used the symbols of multiplication and addition on an explicit way, but the sense of these operations remains intuitive, as he didn't specify the rules they should follow. However, thanks to the analogy with the regular algebraic operations, Galois could do with groups exactly the same thing as what was usually done at that time to introduce the rules of calculus with numbers or algebraic symbols. As a matter of fact, this process by analogy was used by Bézout in the case of negative quantities in his textbook of arithmetic [Bézout, 1781] and by Lacroix in the case of algebraic and imaginaries quantities in his textbooks of algebra

[Lacroix, 1807]. In a word, because Galois used a notation similar to the one used for usual algebraic quantities, he could also use methods and practices that must have sound "natural" for him and for contemporaneous mathematicians.

To sum up the first part of this paper, I would like to emphasize the fact that choosing a way to write the groups is linked to different way to think about this object – it is to say to the very sense of what a group exactly is, of what it is made for, and of what it is made of. It also makes some kind of proofs easier, and other kinds more difficult. For instance, with a table, Galois could look to every elements, ask if some of them would not "fit" together, and use lines and columns to show this point mathematically and visually. With a single letter, he could do "as if" groups were regular algebraic quantities, independently from equations, and look for their specific properties in term of decomposition. But it is also important to notice that Galois had not invented these notations. They were already used at that time for other objects. This is of course a too short case study to draw general conclusions, but it almost raises a question. Symbolism doesn't carry immanent ways of thinking but, on the contrary, some chains/trends of reasoning, which are historically constructed and which are part of specific epistemological cultures, can circulate thanks to them, in particular from one mathematical field to another. In that context, I will now look to this circulation phenomenon a little more closely, analysing the works of some mathematicians who used Galois's published research during the 1850s and 1860s. The underlying question could then be: what exactly circulates while other mathematicians took groups over for their own research?

3. "Groups" in the work of Cayley.

In 1854, Cayley published the two first parts of a paper entitled "On the theory of groups, as depending on the symbolical equation $\theta^n=1$", in the *Philosophical Magazine*. It was continued with a third part, published in 1859. At the very beginning of this paper, just after having given a definition of the word "group", Cayley indicated, in a footnote that "the idea of group, applied to substitutions, [was] due to Galois". The definition Cayley gave for a group was the following:

> A set of symbol,
>
> $1, \alpha, \beta, \ldots$

all of them different, and such as the product of any two of them (no matter in what order), or the product of anyone of them into himself, belongs to the set is said to be a group. It follows that if […] the symbols of the group are multiplied together so as to form a table, thus:

		Further factors			
		1	α	β	..
Nearer factors	1	1	α	β	..
	α	α	α^2	$\beta\alpha$	
	β	β	$\alpha\beta$	β^2	
		:			

that as well each line as each column of the square will contain all the symbols 1, α, β, …". [Cayley, 1854, p. 41].

Thus, in Cayley's paper, there was two ways for writing groups: using a list of elements and using a multiplication table. However, in spite of what Cayley claimed, none of them was close to Galois's symbolism. In fact, what had circulated from Galois to Cayley together with the word "group" seemed not to be symbolism. This raises two questions: first, what are the specificities of Cayley's notations, and, in particular, where could they come from and what kind of mathematical practices are they linked with? Second: what did Cayley exactly take from Galois?

First of all, we have to remark that Cayley used the letters 1, α, β, … to symbolize general operations. Each of them could be, for instance, one of the permutations that Galois wrote with a list of letters. But using one letter instead of a list, Cayley could multiply these operations just like any algebraic symbols; he could also write the result just like an algebraic symbol. In other words, with this notation, he could use the "regular" algebraic chains of reasoning, without having to consider in what these symbol could be special. As a consequence, he could apply to the groups the general ideas that one can read, for instance, in Peacock's *Treatise on Algebra* [Peacock, 1830], or use the same process than the one used by Babbage in his "Essay towards the Calculus of functions" [Babbage, 1815]. In other words, this choice in notations was linked to a mathematical practice that was somehow typical of the mathematical culture that Cayley belonged to, that is to say the one that is often called "the Cambridge algebraic school" [Durand-Richard, 1996]. But this choice in notations was also linked to a specific way to examine the groups. The calculation that one could do over the elements was supposed to lead nearly automatically to the interesting properties of the set:

algebraic calculation on the elements of the groups was the mathematical practice on which the very nature of the groups was founded.

More precisely, Cayley used two notations at the same time to study groups. For instance, when he looked to the groups of four elements, he began using the list-notation to find out what were the different possibilities, and, then completed the list by writing, in parentheses, the restrictions imposed by the definition.

Hence, one of the groups was written:

$$1, \alpha, \alpha^2, \alpha^3 \ (\alpha^4=1).$$

While the other one was written:

$$1, \alpha, \beta, \alpha\beta \ (\alpha^2=1, \beta^2=1, \alpha\beta=\beta\alpha).$$

But after that, in a second time, Cayley associated a table to each of the groups he had found [Cayley, 1854, p. 42-43]:

	1,	α,	β,	γ
1	1	α	β	γ
α	α	β	γ	1
β	β	γ	1	α
γ	γ	1	α	β

	1	α	β	γ
1	1	α	β	γ
α	α	1	γ	β
β	β	γ	1	α
γ	γ	β	α	1

In fact, these tables were not tools to prove the results. They were just a way to *show* them, and to make clear the differences between the two configurations. In that sense, this table notation was very linked to what Cayley really wanted to convince his readers of. As a matter of fact, the point of the paper was to show that "systems of this form are of frequent occurrence in analysis, and it is only on account of their extreme simplicity that they have not been expressively remarked". To develop that purpose, a long further development was dedicated to giving several examples taken from elliptic functions, quadratic forms, matrix and quaternions theories. So, the meaning that Cayley gave to his groups was the one of models with which one could see (in the literal meaning) the similarities between distinct mathematical situations. And, for that purpose, the multiplications tables were very convenient tools, as they provided an easy way to check which configuration of group corresponded to a given particular case.

Besides, this kind of tables was constructed to show how the elements of a given set operated on each other. In that, they were very different of the ones Galois had used, which were constructed to show that some elements of the group could be put together[4]. Therefore, the notations that Cayley used made doable operations that weren't with other ones (and in particular with Galois's). Using lists can be considered as a tool for mathematical proving, and using tables is a way to reach the general aims of Cayley's paper. These ways to write groups also betray the very meaning that Cayley gave to them and, in particular, the fact that he saw them as "algebraic" in the sense of the Cambridge school. By the same token, the notations may influence the reader's understanding of the paper. Using tables that recall the ones that the readers may have seen in other works of the British algebraic school implicitly indicates that symbolic algebra is the right framework to understand this paper.

We could ask ourselves, therefore, why Cayley found it necessary to quote Galois's work or, in other words, what has circulated if it's neither the symbolism nor the very meaning of what is a group? My hypothesis is that the answer can be found in Cayley's writing practices and, more precisely, in the bold lines that we can see in the second table of the group of four elements. As a matter of fact, this bold line separates the four elements 1, α, β, γ in two sets of two elements, 1, α, and β, γ. This could tend to show that, in the particular configuration that is represented by this table, the group can be split into two smaller ones. Hence, this notation could be a way for Cayley to express one of Galois's fundamental characteristics of groups, namely that they can be cut into smaller ones under some conditions. In that particular case, we can note that the symbolism itself has not been imported from Galois's work to Cayley's, but that some "ideas" about groups have in fact been translated from one system of notation to another through the circulation process.

[4] Cayley's notational choices reflect in fact his belonging to the Cambridge algebraic school, as Cayley's tables were very similar to the one used by Hamilton on quaternions [Hamilton, 1853, p. 538], or even to one found in Augustus de Morgan's *Formal Logic* [De Morgan, 1847, p. 74].

4. Kirkman's heterodox group theory

Cayley's paper is now, by far, the most famous reference about groups in English language in the middle of the 19th century. However, Cayley was not the only one to write about groups in the United Kingdom at the time. Another mathematician, Thomas Penyngton Kirkman, had constructed a whole "group theory" in a set of papers he wrote during the years 1860-1862[5]. Kirkman did not adopt Cayley's general viewpoint, and limited his enquiry to substitutions group. However, he followed Cayley when he gave a definition of what was a group according to him, writing that the "usual test" was to check if the product of any two substitutions of the group was a substitution of the group, which was exactly the definition hat one can read in Cayley's paper (quoted above). Actually, Cayley and Kirkman wrote to each other from the second half of the 1840s, and knew each other's works [Crilly, 2006, p.142-154 and 247-250]. In that context, it is not very surprising that Cayley's paper had been quoted several times in Kirkman's works. Kirkman's works on group and his interest for it certainly had to do with the Algebra that was practiced in England at the time: Kirkman had published papers in leading journals like the *Cambridge and Dublin mathematical Journals* or the *Philosophical Magazine* and he had contacts with other British leading mathematicians. However, the reasons of his interest for groups did not come from his local intellectual environment, and the practices he associated to them were quite different from the ones of symbolical algebra. Among the several references that Kirkman made to Cayley, one is of particular relevance to emphasize this point:

> The difficulty of the step from the analytical definition of a group to its actual construction is shown by the fact that M. Cayley did not succeed in constructing this group till long after he had published its definition [Kirkman, 1860, p. 394].

Thus, even if he recognized Cayley's works as highly valuable, it was not his "analytical" approach that Kirkman wanted to follow. According to Kirkman, the default of this method was that it couldn't provide easily the expressions of the groups one was looking for. More generally, this quotation shows to what extend Kirkman's and Cayley's projects differed. Cayley wanted to show that the group concept was "the hidden reason" why some very different mathematical phenomena worked on the same way, whereas Kirkman did seem to have this kind of "meta-mathematical" aim while working on groups.

[5] The many papers published by Kirkman are listed in [Biggs, 1981].

Instead, Kirkman was interested by an important application of the groups, namely finding the number of values a function can take when one permutes its variable. This has been at the time "hitherto quite a French question" [Letter quoted in Crilly, 2006, p. 247]: the problem originated in the works of Lagrange on equations [Lagrange 1770; Lagrange 1826], and had been tackled first by Cauchy, and second by the young mathematicians Serret and Bertrand, respectively in 1845 and 1850 [Bertrand, 1845; Serret, 1850; Ehrhardt, 2012, p. 160-170]. In 1858, it had become the subject of the *Grand Prix de mathématiques* of the French Academy of Science, to be attributed in 1860 and for which Kirkman's longer memoir about groups was competing.

The fact that the issue of substitutions (in particular in their links with the theory of equations and of the number of values of a functions can take when one permutes its variables) was one of the frameworks of Kirkman's papers can be seen in the quotations he made to other mathematical works (those of Cauchy, Galois, Jordan, but also of Betti). It can also be seen from some of the practices he associated to groups, which came from Cauchy's approach to the problem[6]. For instance, after having defined a group as a kind of sum of its elements ($G=1+ A_1+ A_2+\ldots +A_{k-1}$ for a group of order k), Kirkman took from Cauchy the idea of multiplying on the right and on the left (to obtain $PGP^{-1}=1+ PA_1P^{-1}+ PA_2P^{-1}+\ldots + PA_{k-1}P^{-1}$). This allowed him to calculate at the same time with the sets and with its elements, as Cauchy had done with the systems of conjugated substitutions[7].

The Academy received three anonymous memoirs, but finally decided not to give the Prize, because "none of them answered in a sufficient manner to the intents of the Academy". About Kirkman's memoir, the Academy added:

> The memoir used a very clever notation that could certainly provide simplifications in the study of substitutions groups but that, however, it contained very few new and truly important facts [*Comptes rendus hebdomadaires des séances de l'Académie des sciences,* t. 52, 1861, p. 555-556].

[6] On the contrary, Kirkman was very careful in explaining the difference with Galois's work and his own research.

[7] Another instance of practice taken from Cauchy, is the definitions that Kirkman gave of permutation and arrangement. Cauchy had defined the product of an arrangement by a substitution:

$$\begin{pmatrix} xzy \\ xyz \end{pmatrix} xyz = xzy .$$

Kirkman putted it as: $\frac{A_m}{A_n} B = C$, and adding that "the effect on the substitution on B is to exchange in B, for any letter a, that which stands above a in the substitution".

Kirkman felt very angry about that, as one can see from the many times he came back on it in his following papers. However, it seems to me that what was new in Kirkman's papers was maybe less a question of "facts" than a question of methodology. The very fact that Kirkman defined the framework of the Prize as a "French question" shows that he was aware that his own approach might have not been the better one to convince the French academicians. He even explicitly explained that his own methodology was completely different from the one used by the other mathematicians:

> The most remarkable thing in this method is that we need no algebraical substitutions: we are never conscious of their existence. It turns out that the ingenious and learned efforts of the French and Italian mathematicians to conquer this theory with algebra, with its formidable army of congruences and imaginaries, have been from the beginning a brilliant error. The problem is tactical, ad its solution is tactical. [Kirkman, 1862-1863, p. 140]

In other words, even if Kirkman's use of groups and some of the practices he associated to it could be inscribed within this "French tradition", something in the way he dealt with groups remained outside the theory of substitutions. To understand where Kirkman's heterodoxy came from, we have to look to the explicit methodological framework that Kirkman referred to, which he called "Tactics", and which involved specific notations and practices of proving.

Kirkman defined a "tactical investigation" as "one in which no numerical equations or congruence are necessarily used". More precisely, the tactical methods rely on the handling of graphical process, and avoid, if possible, the use of calculation. Hence, tactics has to do with combinatorics, but it has much more to do with practices of handling tables and lists of numbers.

The way by which Kirkman found the square roots of the substitution $\begin{pmatrix} 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$ gives us a good example of what a tactical practice is.

Kirkman starts with two auxiliary groups:

$$\begin{matrix} 1 & 2 & 3 & 4 \\ 2^2 & 1 & 4 & 3^2 \\ 3^2 & 4^2 & 1 & 2 \\ 4 & 3 & 2 & 1^1 \end{matrix} \qquad \text{and} \qquad \begin{matrix} 1 & 2 & 3 & 4 \\ 2^2 & 1 & 4^2 & 3 \\ 3^2 & 4 & 1 & 2^2 \\ 4^2 & 3^2 & 2 & 1 \end{matrix}$$

then he replaces the number 1, 2, 3 et 4 respectively by:

1 2	3 4	5 6	and	7 8
2 1	4 3	6 5		8 7

Each of these tables represents a group of order 2. When a number of the auxiliary group is written with a square, it is replaced by the table symmetric to the preceding one. For instance, the line 2^2 1 4 3^2 becomes:

4	3	1	2	7	8	6	5
3	4	2	1	8	7	5	6

Doing so Kirkman obtains two groups of order 8

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
4	3	1	2	7	8	6	5
3	4	2	1	8	7	5	6
6	5	8	7	1	2	3	4
5	6	7	8	2	1	4	3
7	8	6	5	3	4	2	1
8	7	5	6	4	3	1	2

and

1	2	3	4	5	6	7	8
2	1	4	3	6	5	8	7
4	3	1	2	8	7	5	6
3	4	2	1	7	8	6	5
6	5	7	8	1	2	4	3
5	6	8	7	2	1	3	4
8	7	6	5	3	4	1	2
7	8	5	6	4	3	2	1

The first line is identity, the second one is the substitution that was given, and the others are its square roots [Kirkman, 1860-1861].

This kind of practices of proving could seem very strange today. It also let the French academicians of 1860 stonily indifferent. Kirkman justified the interest of his method by its simplicity and convenience, because, "having your group on the page you [had], in three seconds, in its most simple and useful form for the comparison or for all computations, one of the explicit functions required" [Kirkman, 1862-1863, p. 141]. As a matter of fact, this way to deal with groups, which was not heavily theoretical but softly combinatorial, was linked to a specific use of groups, and to an idea of how and what for this use worked.

First, we can notice that, Kirkman, just like Galois, used his tabular notation as a tool to write his proof, which actually entirely relied on the disposal of the numbers. However, Kirkman's tabular notation did not seem to be a way to show the reader how the proof worked and to make him understand the process. On the contrary, Kirkman's method looks like a "magic recipe", for which it is difficult to get into the reason why the proof works like that and not in another way. Kirkman didn't tell, for instance, how the two first auxiliary groups were chosen.

Second, this process is, just like the ones Galois used with tables, a practice of mathematical proving that avoids calculation but it is also a way to replace it. It is the case, for instance, when Kirkman uses the square symbol to show that he reverses the two lines, which could have been done instead with a substitution. Hence, Kirkman's tables were not only a mathematical way to express mental images that would be difficult to express in words. They also offered a way to write proofs that was alternative to the the algebraic language.

These two characteristics of Kirkman's practice of mathematical proving, which used groups symbolized by number tables, can be related to the specific context from which Kirkman's research took its coherence and meaning. As a matter of fact, Kirkman's interest for "the number of values that a function can take when one permutes its variable" originally came from a kind of mathematical activity that did not took place within universities, scholarly societies or scientific academies, but essentially within popular journals: the one of solving mathematical puzzles. The paper about combinatorics thanks to which Kirkman managed to get in touch with Cayley, in 1846, was a particular case of a more general problem, which is to find the greatest number of combinations of y elements that can be made with x symbols, so that no combination of z elements together shall be twice employed. This problem had been set two years previously, in 1844, as the Mathematical Prize Question of the *Lady's and Gentleman Diary*. Hence, the interest of Kirkman for combinatorics came from a challenge in a popular annual publication [Biggs, 1981]. When solving the problem with y=3 and z=2, Kirkman used processes very similar to the ones he would call "tactical" ten years later: writing down the objects with a particular layout, using columns and circles, and presenting the procedure to follow as the "rule" to solve the problem [Kirkman, 1847]. The fact that Kirman was not very concerned with explaining why his process worked might have had to do with this "recreational context", where the most important thing is to get the result, the "value" associated with what a good proof being in that case certainly simplicity and convenience – that is to say "cleverness".

After having published his paper in the *Cambridge and Dublin mathematical journal* in 1846 Kirkman came back to recreational mathematics, setting a mathematical puzzle – known as "the fifteen schoolgirls problem" – which is a particular case of the problem above with x=15 and which was solved by Cayley himself. This famous problem was the subject of a priority quarrel between Kirkman and James Joseph Sylvester, another mathematician close to Cayley. At the same time when Kirkman published his memoir on group theory, Sylvester wrote several papers in order to defend his paternity [Sylvester, 1861a; Sylvester, 1861b; Sylvester, 1861c; Sylvester, 1862]. In these papers, he defined Tactics as the "third pure mathematical science", whose object was order while the two others had number and space for objects, and explicitly mentioned the theory of Groups as being one of the special branches of Tactic. Our point, here, is not to come back on this quarrel; but what is important to notice for our purpose is, first, that the practices that Sylvester qualified as “tactical” where very similar to the ones of Kirkman related above; second, that, Cayley, Sylvester and Kirkman where writing to each others at the time, the research about groups being one of the interests they shared. Hence, the fact of using groups in a tactical way to avoid algebraic calculation was not only a pragmatic way to practice mathematics: it was also the very aim of the bigger field, Tactics, in which Kirkman, just like Sylvester, inscribed his research on groups.

5. Dedekind’s Galois theory

Richard Dedekind made a seminar about Galois’s works during the years 1856-57 and 1857-58 at the University of Göttingen [Dedekind, 1981][8]. In that occasion, he also gave explanations about the notion of group. This field of research was at the time quite new for him. The previous papers that he had published, in 1853 and 1855, were about Eulerian integrals and rectangular coordinates. It’s quite obvious that Dedekind knew the Galois’s paper published in *Liouville’s Journal* in 1846, but we can assume that he knew Cayley’s one too. As a matter of fact, in one of his previous papers [Dedekind, 1855], Dedekind quoted a memoir by Boole that had been published in the same issue of the same journal that Cayley’s paper on groups. Hence Dedekind had two different approaches of groups at his disposal when he started his seminar. The beginning of the manuscript of his lectures shows that he actually used both of them. That Dedekind was a reader of Galois can be seen in the very order of his lessons, which follows Galois’s *Mémoire*, and in the recurrent quotation of Galois’s name. A consequence of that reading was that Dedekind only considered groups of substitutions, which was sufficient in the framework of the theory of equations. On the other hand, Dedekind gave a definition of groups, which Galois had not done, one close to Cayley’s:

> A set G of g substitutions is a group of order g if every arbitrary product of substitutions contained in G is still contained in G. [Dedekind, 1981, p. 64].

Moreover, even if Dedekind’s primary focus was on substitutions, he explained that:

> The following research is based only based on the two previous fundamental results and on the fact that the number of substitutions is finite. Therefore, the results remain valid for any finite set of elements, objects or concepts that would satisfy [these rules]. […]

[8] The notes he took remained unblished untill 1981.

> We will keep the notations of the theory of substitutions because it is simpler, but we will also use the more general conception in what follows. [Dedekind, 1981, p. 63]

Yet, Dedekind took from Cayley the definition of a group and the idea that the elements could be anything; but he didn't keep Cayley's notations. Moreover, Dedekind wrote that he would use the notations of the theory of substitutions, which makes one expect that Dedekind would write tables of elements to represent the groups, as Galois did, or at least that he would follow the ideas that Cauchy had developed in that field in 1844. However, always used the single letter notation to represent the groups. In other words, he precisely chose the notation where the substitutions framework of the group concept was the thinner. Doing that, he actually erased any specificity that substitutions could provide to the group concept.

Moreover, this choice in symbolism had other consequences. As we have already seen in the case of Galois's research, it follows from the single letter notation that one can consider each group as one mathematical object, and not as a cluster of elements. In fact, using this single letter notation, Dedekind didn't do calculation *within* the group, but *with* the group. Just like Galois and Cayley, Dedekind took his inspiration from another mathematical field. But instead of transposing the procedures of algebraic calculation, he used tools coming from number theory[9]. In that context, and contrary to what Galois used to do, Dedekind didn't need a graphical representation to show that a given set was actually a group: he just needed to come back to the definition, and to prove that it was verified. In other words, he associated to Galois's single-letter notation a kind of practice of proof that Galois himself didn't use.

An example taken from Dedekind's manuscript illustrates very well how the mathematical practice associated to one notation can be transposed from one object to another. As a matter of fact, the single letter notation gave Dedekind the possibility to employ the tabular notation as a method of proof, just like Galois did, but, this time, the whole group played the same role as the elements in Galois's work:

> If one forms, under the same hypothesis that prop. V [K is a divisor of G], the following schema:
>
> $$K, \qquad K\theta_1, \qquad K\theta_2, \ldots \qquad K\theta_{h-1}$$

[9] For instance, he defines the "divisors" of a group, the greatest common divisor of several groups, and look for the consequences of these definitions, in terms of divisibility on the number of elements. This is also from the number theory (and more precisely from Euler, but I haven't managed to find exactly where in Euler's works…) that he justified a notation he would nearly systematically use in the proofs [Dedekind, 1981, p. 65]: $G=K+K\theta_1+K\theta_2+\ldots +K\theta_{h-1}$. In the same way, Dedekind combined this calculation on groups with reasoning inspired by number theory and questions of divisibility to prove the theorem II of the first part of Galois's *Memoir*, which explains what happens when one adjoins a quantity.

$$\begin{array}{llll} \theta_1^{-1}K, & \theta_1^{-1}K\theta_1, & \theta_1^{-1}K\theta_2, \ldots. & \theta_1^{-1}K\theta_{h-1} \\ \theta_2^{-1}K, & \theta_2^{-1}K\theta_1, & \theta_2^{-1}K\theta_2, \ldots. & \theta_2^{-1}K\theta_{h-1} \\ \ldots\ldots\ldots\ldots & & & \\ \theta_{h-1}^{-1}K, & \theta_{h-1}^{-1}K\theta_1, & \theta_{h-1}^{-1}K\theta_2, \ldots. & \theta_{h-1}^{-1}K\theta_{h-1} \end{array}$$

Then the sets of all the substitutions contained in any horizontal or vertical line is equal to the group G. [Dedekind, 1981, p. 66]

Thus, while Galois used this tabular representation to show how some elements could be combined to make a group, Dedekind uses exactly the same kind of table to show how some groups could be combined to make another one. With the single-letter notation, groups are manipulated just like if they were algebraic symbols. More precisely, Dedekind made with groups exactly the same thing that Galois had previously done with the *elements* of the group. He thus used a specific way of calculating, or maybe it would be better to say of manipulating, that he imported from Galois's work. He applied it to an object that he also took from Galois, but, in Galois's work, this object and this mathematical practice were not associated.

So, in Dedekind's case, this was not the notational part of the group concept that had been re-elaborated through the circulation process. On the contrary, Galois's original symbolism had been incorporated in a framework that goes from number theory to the idea that the right way to characterize an object is neither to write it nor to represent it, but to define it by advance with general properties that are suppose to show its "real nature". This may be the reason why Dedekind's reading of Galois text is often said to be more abstract than the ones of other mathematicians. But we should also remember that the manuscript with which historians work was written in the particular context of a seminar, which took place in Dedekind's office with only two students. This means that this text may not represent what actually happened during the lecture. Dedekind could have given a lot of further explanation by oral, just answering questions. The final text may appear to be so abstract because details, and in particular calculations, may not have been reproduced in it. Moreover, Dedekind's choice for proofs and notations could be linked to the knowledge and habits of his students. He could have adapted his lessons to them. In that case, Dedekind's reading of Galois' works would be not only correlated to Dedekind's agency and mathematical preferences, but also to the institutional environment in which it this seminar took place.

Conclusion

This case study of some of the notations that were used to manipulate groups shows that they can be linked to very different ways to think about these objects and to use them for proving theorems. Then, symbolism appears to be a good way to put light on the different images and practices of mathematics that are two often hidden behind the same category of "abstraction".

I also want to emphasize the point that symbolism has some kind of autonomy during the circulation of knowledge process: a mathematician may use the notation of another one without transposing neither the methodology of the original paper nor its theoretical framework. Nevertheless, this doesn't mean that this autonomy would be a transcendent property of mathematical symbolism. On the contrary, I would say that it strongly relies on the historical and social contexts where each new use of the original text happens.

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